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Qualitative Calculus of Systems: Exploring Students' Understandings of Rate of Change and Accumulation in Multiagent Systems

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Abstract

Our everyday world is characterized by quantitative change – from fluctuating global temperatures and shifting medical insurance costs to the changes in a car's tire pressure from winter to spring. Often, these quantities do not reflect only a single entity or action, but many different interactions and behaviors. This paper investigates how students think and talk about patterns of quantitative change over time while they interact with a computational agent-based model of population growth, which represents change in population as the result of many entities (simulated people) contributing *individually* to a single changing quantity (population). We found that students often mixed not only mathematical, but also scientific and everyday explanations to make sense of patterns of change. These combinations of explanations led some students to experience (or resolve) difficulties in describing what *rate of change* reflects in the specific case of population growth; and in understanding and interpreting quantitative change in terms of the individual and population-level behavior of a dynamic system.

QUALITATIVE CALCULUS OF SYSTEMS: EXPLORING STUDENTS' UNDERSTANDING OF RATE OF CHANGE AND ACCUMULATION IN MULTIAGENT SYSTEMS

1. Introduction

Using the ideas of mathematical change to understand, navigate, and predict phenomena in our world is becoming an important life skill (Roschelle, Kaput & Stroup, 2000). But the change that we encounter in the world – employment rates, climate change, financial trends – often involves complex interactions between many actors and events. Understanding these phenomena and making sense of them is a matter of understanding how that change is reflective of, generated by, and has implications for, individual entities in a system. This can help prepare students as active and informed citizens, provide a new access point to more formal mathematical topics such as calculus (Nemirovsky, Tierney, & Ogonowski, 1993; Stroup, 2002), and provide a better foundation for students entering the natural and social sciences where such systems are especially common (AAAS, 1991).

We argue that understanding the relationship between individual behavior and population-level trends is of particular importance when making sense of systemic change as it manifests in real-world contexts. Reports of changes in unemployment rates affects individual workers, and influences the ways that individuals decide to spend their money. The relationship between changes in a quantity and what it means for an individual member of that system may be unexpected and even seem contradictory: consider, for example, what while slowing job loss is considered a good economic indicator, it also reflects that an individual is still more likely to *lose* her job than *find* one. Similarly, although a population increases at an increasing rate, the reproductive behavior of each individual in the population does not need change (in fact, population may still increase as individual reproduction falls!).

From this perspective, the mechanisms that generate change in systems need to be understood at a different level than the change itself – for example, change in a population is the result of each individual's reproductive behavior, rather than the result of an overall population growth rate. We are interested in the potential for *agent-based modeling*, a way of simulating the behavior of individual agents to derive overall patterns of system behavior, to allow students to explore quantitative change in complex systems. Using this method, quantitative change can be encoded and represented as the aggregation of individual behaviors over time – for instance, unemployment can be represented in terms of the combined interactions between firms, employees, and consumers; or population dynamics as the result of individual reproductive and predatory behavior. Understanding change in such systems, however, may present new challenges and leverage different student understandings and experiences than those they employ to make sense of simple systems. This study is an effort to explore those questions as part of a larger agenda to develop tools and interventions that provide students with opportunities to build, explore, analyze, and interpret models of quantitative systemic change in the real world.

In the following sections, we leverage existing literature and recent advances in mathematics education and the learning sciences to describe some key aspects of understanding change in systems. We then motivate the potential for agent based modeling to provide students a rich environment within which to explore systemic aspects of change, and use the example of simple population growth to describe how agent-based modeling makes explicit many of the aspects we believe are important in understanding change in systems. We present our research questions for this particular investigation, and describe the development of our theoretical and analytic framework. Finally, we describe and present the results of our study, in which 11 high school students were interviewed about their thinking around the quantitative patterns produced by an agent-based model of simple population growth.

Our findings suggest that in addition to graphical and mathematical ways of reasoning already identified in prior work, students employed a variety of additional representational (model-based) and context-specific (group- and individual-behavior-based) resources when reasoning about an agent-based model of exponential population growth. Students were adept at answering questions related to describing rate of change and accumulation in this context, and were able to interpret the model in terms of both the exponential function and in terms of simulated population growth. However, many students experienced difficulty (a) coordinating agent- and aggregate-level behavior by relating trends in population growth with individual reproductive behavior, and (b) interpreting rate of change as representing change in population. We present several cases where students move toward these understandings by explicitly considering the agent-level rules of the model, and by noticing and making sense of unexpected features of the model (such as noise in graphs). We argue that these two links: from agent- to aggregate-level behavior, and from aggregate-level change to trends over time, comprise important building blocks for understanding change in systems.

1.1 Extending "Qualitative Calculus" to Complex Systems

There has been a great deal of research on students' understanding of the mathematics of change – primarily in the context of calculus education (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002), the calculus reform movement (Ganter, 2001), or in differential equations education (Rasmussen & King, 2000). For the purposes of this paper, we limit our review to studies that concentrate on a somewhat broader goal. Over the past 20 years, a great deal of research has shown that even young students are able to understand notions such as rate of change, the mathematics of change and variation, or what has been termed a "qualitative calculus" (Piaget, 1946:1970; Nemirovsky, Tierney & Ogonowski, 1993; Kaput, 1994; Stroup, 2002; Confrey & Smith, 1995) – that is, they are able to successfully reason about rate of change and accumulation, as well as the relationship between the two, without relying on algebraic manipulation or traditional ratio and proportion-based methods. While this work has illuminated the nature of these initial understandings and establishes a number of experiences and environments that can support the development of the mathematics of change and variation (MCV; Roschelle, Kaput & Stroup, 2000) in students, it has largely only been explored in the contexts of motion and other systems for which the mechanisms of change are relatively simple.

This work suggests that students may have ratio-independent (in the sense that rate can be expressed as a ratio of change over a specific unit[1]) notions of change: whereby they are aware that two quantities vary together, or that something is changing more quickly or slowly, but do not assign or identify a unit-based correspondence for the relationship (Confrey & Smith, 1994; Nemirovsky, et al., 1993; Thompson & Thompson, 1996). These notions include ways of *how much* a changing quantity is, *how fast* that quantity is changing, and the "reversibility" or relationship between that quantity and its rate of change (an informal description of the integral, derivative, and the Fundamental Theorem of Calculus; Stroup, 2002). They suggest that this qualitative understanding of *change* and *motion* can include many of the same conceptual elements as more formal calculus (Thompson, 1994), and of students' notions of rate of change at more formal levels (Carlson, et al., 2002; Carlson, Jacobs, & Larsen, 2001).

How might these concepts translate to change in multiagent systems? Certainly these same notions – rate of change, accumulation, and the correspondence of the two – play an

[1] Or at an instant, in the case of formal calculus.

important role in one's understanding and interpretation of quantitative change in a complex system. The behaviors that each of those concepts represent, however, and the ways that they manifest exist at different levels: agents in a system can *individually* contribute to change, often in a number of ways (for instance, in the case of population, by giving birth, dying, and so forth). In this sense, we can think of change in complex systems as aggregation first across behaviors within an individual, then across individuals within a time-step (comprising the rate of change), and finally, across time.

Figure 1. Quantitative change in complex systems is caused by multiple individuals or components, across multiple periods of time.

1.2 Computational Modeling to Learn the Mathematics of Change

A number of computational and other learning environments have been created that help students to develop this sense of qualitative calculus by exploring change without algebra or calculus – in a number of linked environments that include multiple representations of change such as a generating behavior (such as the motion of an object or individual), a time-series plot, number tables, and so forth (e.g. Trips; Clements, Nemirovsky, & Sarama, 1995; Interacting Diagrams; Confrey, Maloney, & Castro-Filho, 1997; SimCalc MathWorlds; Roschelle & Kaput, 1996). Indeed, graphs (specifically, time-series plots of changing quantities and at times their rates of change or accumulations) have become one of the central elements of study in both the teaching and researching of precollegiate and collegiate-level students, and appear to play an important role in students' understanding of MCV. Specific design elements and activities that have been shown to help students use graphs to make sense of MCV include *dynamically linking* multiple representations of change, enabling learners to *control* the real or simulated phenomena that generate quantitative change (Kaput, 1994; Wilhelm & Confrey, 2003), and interpreting those patterns in terms of specific plot intervals, shapes, and events (Nemirovsky, 1994; Schwartz & Yerushalmy, 1995; Yerushalmy, 1997).

Some educational technologies also exist for exploring change at the systemic level: notably system dynamics modelers such as STELLA (Steed, 1992) or Model-It (Jackson, Krajcik & Soloway, 2000). These environments allow users to identify a number of causal relationships between quantities (for example, the relationship between a population and its reproduction rate, and how that rate contributes in a loop again to population). However, while these modeling systems have been shown to help students understand the causal links and complex nature of many dynamic systems (Doerr, 1996), they do not enable students to explicitly connect patterns of change to the behaviors of the *individuals* that ultimately comprise that system.

1.3 Agent-Based Modeling

In an agent-based model, simulations are created by assigning behaviors (in the case of our population growth model, this behavior is that each individual has small percent chance of reproducing) to multiple individual *agents*. The model executes those behaviors repeatedly, and quantitative trends emerge over time by measuring values after each "tick", or execution of agent-level behavior. By interacting with agent-based models, students can reconcile behavior and outcomes on multiple levels of a system (Wilensky & Resnick, 1999), and develop generative and deep understandings of complex systems in a number of domains (such as biology, Wilensky & Reisman, 2006; chemistry, Levy, Kim, & Wilensky, 2004; materials science, Blikstein & Wilensky, 2006; and physics, Sengupta & Wilensky, 2008). Little is known, however, about how agent-based models can be used to expose students to notions of quantitative change more generally.

1.4 Research Objectives

In this paper, we investigate students' descriptions of rate of change, accumulation, and the relationship between the two in the context of population growth as simulated using an agent-based model. Importantly, this model defines a *per-agent probability* of reproduction, rather than calculating the number of individuals that will be born using the total population. We seek to better understand whether and how students can successfully leverage a "qualitative calculus" to make sense of the mathematics produced by such an agent-based simulation. Our specific research questions for this study include:

1. How do students talk about quantitative change in the context of multi-level complex systems?
2. How do students describe the mechanisms that produce rate of change and accumulation in the context of these models?
3. What are points of difficulty in making sense of quantitative change in the context of these models?

2. Theoretical and Analytic Framework

Our study and analysis leverages two major theoretical perspectives in science and mathematics education: First, that a deep and generative understanding of the calculus of systems can be fostered by thinking of those systems in terms of multiple *levels* (Wilensky & Resnick, 1999) – whereby macro-level mathematical quantities and trends over time emerge from micro-level behaviors. Second, that even students without a background in calculus or ratio and proportion can develop a basic understanding of the underlying concepts of calculus, and that this understanding can be used to reason productively about sophisticated mathematical patterns (Nemirovsky, Tierney & Ogonowski, 1993; Kaput, 1994; Stroup, 2002). We also adopt the broader underlying epistemological perspective of *knowledge-in-pieces* (diSessa, 1993) in our treatment of student reasoning – specifically, we do not seek the presence or absence of specific coherent patterns of reasoning, but rather the employment and coordination of a variety of co-existing, perhaps even contradictory, knowledge resources.

In this section, we briefly review each of these perspectives, and conclude by presenting the framework we used in developing the interview protocol for this study. In our results, we present a revised framework that we used for analysis, which was adapted from this initial framework to include a number of new categories and considerations introduced in the data.

2.1 Thinking in Levels and the Mathematics of Systems

Most work in student understanding of qualitative calculus is powerfully rooted in students' understanding of, and embodiment of, motion. This makes sense: distance and speed are easily perceived quantities, and even acceleration can be thought of in embodied and intensively quantifiable terms. But many of the systems that are studied in science, technology, engineering and mathematics (STEM) and the social, behavioral, and economic sciences (SBE) using these core ideas are not simple: many components, individuals, or other elements of a system and their behaviors all contribute to the same measured rate of change. For example, each atom in a gas affects the total measure of that gas' pressure and temperature (Wilensky, 2003), and the reproduction and predation patterns of each individual in a predator-prey system all contribute to the same population measures (Wilensky & Reisman, 2006).

The importance of these agent-level entities and behaviors in producing aggregate-level measurable quantities is an instance of what Wilensky and Resnick term "thinking in levels" (1999). By reframing a number of scientific and mathematical phenomena as multi-level systems, understanding and analyzing these systems can become accessible to many more students (as has been already shown in a number of domains; such as biology, Wilensky & Resnick, 1999; Wilensky & Reisman, 2006; chemistry, Levy, Kim & Wilensky, 2004; probability, Abrahamson & Wilensky, 2004; materials science, Blikstein & Wilensky, 2006; and physics, Sengupta & Wilensky, 2008; Wilensky, 2003) We believe that mathematical models, too, can be productively conceptualized as multi-level systems: wherein which aggregate level properties can be described in terms of the micro-level behavior that affects and changes those properties. Therefore, we believe that understanding how students make sense of the mathematics of systems requires an explicit attention to the *levels* at which students attribute quantities, and changes in quantities.

2.2 Extending Qualitative Calculus: How Fast, How Much, and for What?

In a synthesis of the large body of research into students' understanding of the mathematics of change, Stroup (2002) argues that students can develop a "qualitative calculus" as a cognitive structure in its own right that is different from (but can support) traditional conceptions of calculus based on ratio or proportion. In developing this structure, learners identify rate as an *intensive* quality (that can get faster or slower), learn to differentiate between *how much* and *how fast* an object travels, and to coordinate the relationship between those two quantities – in essence, developing an early qualitative form of the Fundamental Theorem of Calculus. As a part of this review, Stroup identifies some specific ways that learners explicate rate in terms of intensive quantities (specifically, their intensification of *how fast*) and their ability to establish a relationship between *how much* and *how fast* by identifying features of graphs such as "slantiness" and direction.

In this study, we are particularly interested in how students' explanations and interpretations of the mathematics of complex systems are similar to, and depart from, these known ways of talking about rate of change, accumulation, and the relationship between the two. There is some evidence that a well-established understanding of these concepts can transfer to non-motion contexts, although these contexts are not complex systems (Wilhelm & Confrey, 2003; Herbert & Pierce, 2008). It would be particularly interesting to see whether students are more likely to speak of the mathematics of complex systems in terms of motion/speed, graphical features, or some other way.

2.3 The Preliminary Qualitative Calculus of Systems (QCS) Framework

Our interview design and preliminary coding schematic allowed for each student response to be coded by concept dealt with, metaphors and explanations used while describing that concept, and the presence or absence of a context-specific explanation at the agent or aggregate level.

CONCEPT	HOW FAST: the rate of change of the population
	HOW MUCH: the population itself
	COORDINATION: how rate of change is related to the population itself
METAPHORS AND EXPLANATIONS	VISUOSPATIAL: visual features of the model, such as “person density” or proximity of children to their parents
	QUANTITY: numerical quantities provided by the model
	GRAPHICAL: graphical features, such as height of a graph or its “slantiness”
	MOTION/SPEED: terms associated with motion or speed, such as “faster” or “closer”.
CONTEXT-SPECIFIC EXPLANATION	AGENT-LEVEL: what a single agent in the model is experiencing, such as “each agent has a chance of reproducing”.
	AGGREGATE-LEVEL: a population-level description of behavior, such as “there are more agents to make more babies”.

Figure 2. The Preliminary Qualitative Calculus of Systems Framework

Guided by this framework, our semi-structured interviews specifically focused on students’ reasoning about population levels, change in population, and the relationship between the two graphs featured in the model. Within each of these broad foci, questions and probes focused on students' reasoning not only about how they arrived at their answer, but also why they believe it to be true, what it means in terms of the specific case of population growth, and what it means in terms of agent- and aggregate-level behavior in the system. A full copy of the interview protocol is included in the appendix.

3. Methods

We interviewed twelve junior and senior high school students (8 males, 4 females) enrolled in an accelerated precalculus summer program at a large, high-performing public Midwestern high school. Students were familiar with notions of rate of change and the derivative, and were actively discussing these concepts in class. Each student was interviewed for 30 – 45 minutes. Interviews were videotaped with two cameras (one to capture student interaction with the computer, the other to capture the student's face and gestures) and transcribed. One interview (male) is not included in this analysis because portions of the interview were lost due to equipment malfunction. Gestures toward the computer screen made during the interview were also included in the transcript and analysis as evidence of reference to specific graphs or other model features. The first author coded transcripts; the final coding scheme is included in the next section.

During the interview, students were first presented with an agent-based model of simple population growth, which begins with 50 "agents" each with a 1% chance of reproducing at each unit of time. The rules for the computational model were explained to students, and then the model was run for 150 "ticks", or repeated executions of the model's rules. Students were asked to explain and interpret a plot of the population produced by the simulation over time, a plot of the number of people born each time unit, and the relationship between the two graphs. Next, students were invited to modify the model to add additional behaviors (such as death, conditions for reproduction, and so on), to predict the resulting mathematical trends, and to explain the mathematical trends that manifested when the model was run. In this paper, we only discuss the first portion of the interview.

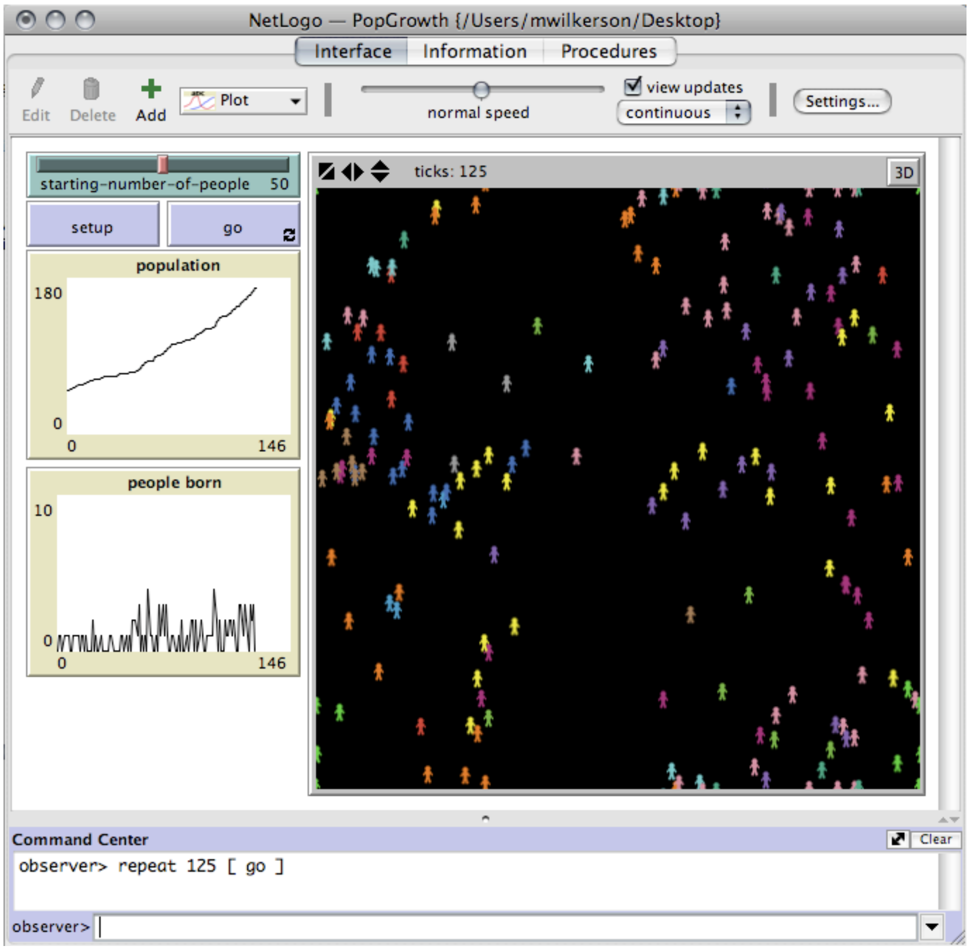


Figure 3. Participants were shown a simple agent-based model of population growth

4. Results

We present our results in two sections: first, we introduce our revised framework that describes the key concepts and ways of reasoning that students used when making sense of systemic change in the context of population growth, in an attempt to broadly explore our first two research questions, Q1. *How do students reason about quantitative change in the context of complex systems?, and* Q2. *How do students describe the mechanisms that produce rate of* *change and accumulation in the context of these models?* Next, we will begin to address Q3. *What are points of difficulty in making sense of quantitative change in the context of complex systems?*, by highlighting two specific areas of difficulty in some students' making sense of population growth: the relationship between agent- (individual) and aggregate- (group) behavior, and the relationship between model behavior and the rate of change of population growth as exhibited in the model. Some students experienced *slippage between levels* (Wilensky & Resnick, 1999) when making sense of exponential growth in terms of unchanging individual level behavior, and some were not able to make sense of the relationship between people born per year and notions of rate of change of the population. In the context of section 2, we also begin to explore how students who experienced these difficulties began to resolve them by paying attention to and coordinating specific features different representational forms: specifically the model plots, the programmatic behavior of the model, and their own knowledge of the mathematical notions of rate of change and accumulation.

4.1 Reasoning about Change in Complex Systems

In Section 2.3, we introduced a preliminary framework for a "Qualitative Calculus of Systems", which included the concepts (rate of change, accumulation, and their relationship), metaphors and explanations (such as graphical features, numerical trends, visuospatial cues, or motion metaphors) for making sense of those concepts, and context-specific explanations (such as the behavior of an individual or group). In this section, we begin by presenting a modified version of this framework that includes a number of elements that emerged from our analysis of the interviews. We then use this framework to paint a broad picture of students' reasoning about change in complex systems as they answer questions about population levels and rates of change in the context of an agent-based model of exponential growth.

4.1.1 A Revised Qualitative Calculus of Systems (QCS) Framework

After reviewing our student interviews, we revised our preliminary QCS framework in a number of ways. First, we noticed that rather than treating context as an additional or merely interpretive dimension of understanding, students used contextual explanations in the same way as they used other mathematical resources and ways of explaining their responses. For instance, students could respond that they knew population was highest at the end of the model run because the graph of population was highest at that point, or because they knew that people could only be born and not die in the model and so population would always get higher:

> **[*Int 6: Graphical*] Um, because it's on the y-axis?**
> **[*Int 1: Programmatic*] Because each tick population can't decrease only increase […] you can never get lower so more ticks, more population.**

In other words, there was not a clear distinction between context-specific and mathematical explanations, and they were often used together or interchangeably. Therefore, we consolidated these categories into a single category called “information sources”, with context-specific explanations (that is, explanations that leverage the behavior of individuals or the group) included under the subheading “behavioral”.

Second, students cited a number of information sources and reasons for their responses to questions about population growth and rate of change that were not well accommodated by the preliminary framework. These included referring to their knowledge of properties of the exponential growth function specifically (FUNCTIONAL), their understanding of the behavior of the computational program (PROGRAMMATIC), and their general understanding of population growth as a systemic phenomenon (SYSTEMIC):

> **[*Int 6: Functional*] Um, because it's increasing exponentially, so um, I don't know how to explain it (it's okay), but um yeah. I don't know, but it is increasing exponentially**

> **[*Int 2: Programmatic*] It doesn't, it doesn't decrease, it's just increasing (so it's just) so the latest tick (okay) is the highest.**
> **[*Int 3: Systemic*] …it's just like any other population growth I guess…**

Finally, we noticed that talking about systemic quantitative change introduced a number of new concepts that are not typically dealt with when considering change generated by only one entity or behavior – concepts that some students employed to explain anomalous or noisy features of the graph. These concepts include *stochasticity* or the unpredictability and randomness of behavior in many such systems, the *heterogeneity* of different entities in a system, and the *aggregation* of multiple behaviors that contribute to a single measured value. In other words, a quantity changes not only as a result of predictable patterns and behaviors but also as a result of *random* behavior, it can mask the *heterogeneity* of the population it models, and it can change as the result of more than one behavior (for instance, population changes due to the *combined influences* of birth, death, immigration, and so forth):

> **[*Int 3: Randomness*] Because if they only have a .01 chance of reproducing, it doesn't mean they're gonna be doing it every second.**
> **[*Int 9: Heterogeneity*] …cause people may not choose to have kids. (mhm) so. (okay) yeah. so some years there might be less and some years there might be more.**
> **[*Int 1: Aggregation*] Um, so I guess people can die in this model?**

Below, we introduce a revised framework that includes these new elements highlighted in grey.

<table>
<tr><th>Concept</th><th colspan="2">Information Sources</th></tr>
<tr><td>How Fast: the rate of change of the population</td><td colspan="2">Visuospatial: visual features of the model, such as “person density” or proximity of children to their parents</td></tr>
<tr><td>How Much: the population itself</td><td colspan="2">Quantity: numerical quantities provided by the model</td></tr>
<tr><td>Coordination: how rate of change is related to the population itself</td><td colspan="2">Graphical: graphical features, such as height of a graph or its “slantiness”</td></tr>
<tr><td rowspan="3">Stochasticity: reference to the unpredictability/randomness of behavior</td><td rowspan="2">Behavioral: terms that reference the behavior of simulated agents</td><td>Agent-Level: what a single agent in the model is experiencing, such as “each agent has a chance of reproducing”.</td></tr>
<tr><td>Aggregate-Level: a population-level description of behavior, such as “there are more agents to make more babies”.</td></tr>
<tr><td colspan="2">Functional: prior knowledge regarding the approximate function</td></tr>
</table>

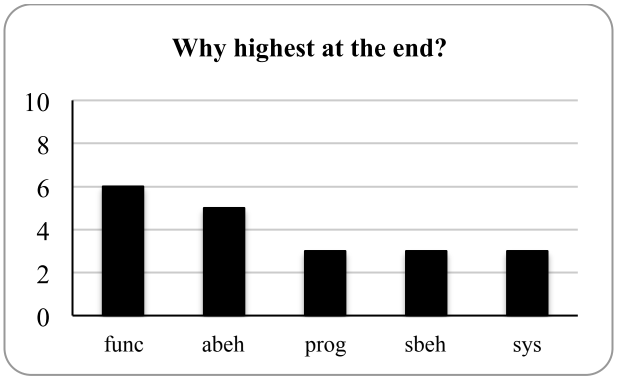
	produced, such as "exponential curves always increase in rate"
HETEROGENEITY: reference to the individuality of/differences between agents	PROGRAMMATIC: reference to the model's coded behavior, such as "the reproduce function makes more of them"
AGGREGATION: reference to multiple simultaneous influences	SYSTEMIC: reference to the general behavior of systems like the one modeled, such as references to bacterial reproduction or "in general population growth…"

Figure 4. A Revised Qualitative Calculus of Systems (QCS) Framework

4.1.2 Reasoning about Rate of Change and Accumulation in Complex Systems

Using this revised framework, we next describe how participants answered questions about the population and its rate of change, and how they explained their answers. We find that in general, while most students use similar strategies to answer these questions (overwhelmingly, by referring to features of the population graph), their interpretations of the graph and descriptions of *why* population and rate of change manifested the way that they did were varied and drew from multiple different resources. Our main conclusion from this portion of the analysis is that participants use a variety of ways to reason about the causes and explanations for change in models of complex systems; sometimes in "piecemeal" ways that are inconsistent. Later, we will argue that some of these different ways are more or less consistent with one another; and that depending on students' preferred resources and what representations and resources they have access to, they can more easily link agent-level, behavioral, and over-time trends.

Reasoning about population. When asked when population was the highest in this model, all students correctly answered that it was "at the end", or by identifying the last time unit of the simulation. 9 out of 11 participants explicitly referred to the plot of population to answer the question. The other 2 more vaguely suggested that it would be toward the end of the simulation: ("at the end" [12], "when the ticks keep increasing" [3]). When asked to reason about why the population would be highest at the time that they indicated, 7 of the 11 participants cited more than one reason for the pattern. They most frequently referred to the properties of exponential functions themselves ("in an exponential growth as time increases the amount of people increases" [10]) or used aggregate-level behaviors ("there are more people well they'll have more babies" [5]) to explain the trend.



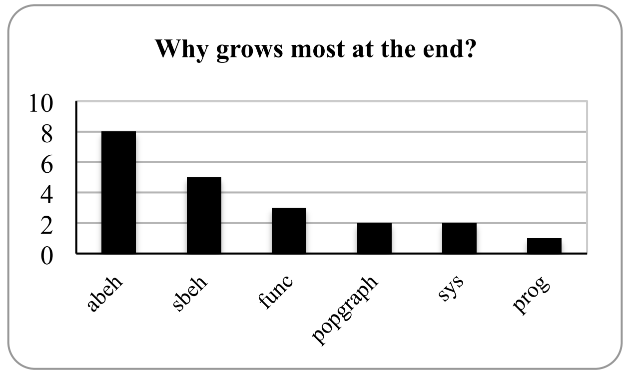


Figure 5. (*left*) Why is population highest at the end? (*right*) Why does population grow the most at the end?

Only two students [4], [7] referred to both agent-level and aggregate-level behavior, and both experienced difficulty or introduced conflicting information when using both levels of behavior to justify the exponential trend. We will discuss this in more depth in the next section.

Reasoning about rate of change. When asked when population changed the fastest, again 9 out of 11 participants explicitly referred to the plot of population, noting that slope indicated rate of change. The other two suggested that rate of change was highest at the end without referring explicitly to the plot; "probably the next time you go" [9], "it starts off s-slower because there's less people" [4]. Only student explicitly referred to the graph of individuals born, as well, noting that it reflected change in the population [1]. Indeed, many students had difficulty explicitly identifying the number of people born as defining or otherwise being related to the rate of change of population in this model. When asked why the rate of change of the population was highest at the end, most participants cited aggregate-level behavior ("the more people there are the more ticks there are means the more people, means the uh chances of reproduction in the entire population are increased" [3]) and individual-level behavior ("if there's more people each of those new people um that were reproduced they could reproduce" [6]) – often together.

Notice that more students cited individual-level behavior (sbeh) as a justification for an increased rate of change, rather than as a justification for an increased population. This makes sense, since the rate of change from one time unit ("tick") to the next is entirely driven by individual-level behavior, whereas population is a measure of the aggregate group. It also seems that reasoning about rate of change, rather than about population trends per se, made the relationship between individual and aggregate-level behavior more apparent. All 5 students that cited individual-level behavior when explaining the cause for an increasing rate of change also referred to aggregate behavior (abeh). Of those 5 students, four were able to clearly explain the relationship between individual-level and aggregate-level behavior:

[Int 1] "There's more people to have that 1% chance."
[Int 2] "there's more people with the same chance"
[Int 6] "if there's more people each of those new people um that were reproduced they could reproduce"
[Int 10] "as there's more people, while they only have a 1% chance of people reproducing, there's more of them"

4.2 Difficulties in Understanding and Interpreting Quantitative Change in Systems

In this section, we will describe two particular types of difficulties that we believe are consistent with difficulties in moving from agent- to aggregate-level notions of change (*slippage between levels*), and from the notion of change to the accumulation of that change over time (*the fundamental theorem of calculus; reversibility).* In each case, we present a few case studies instances and then discuss the turns that the participants and interviewer took to overcome each difficulty.

4.2.1 Reconciling Individual Behavior with Overall Change

It was rare that participants referred to individual-level behavior unprompted as an explanation for the mathematical trends that we discussed during the interview. When they did, it was often in the context of rate of change (as discussed in Section 4.1.2), and in combination with aggregate-level behavior. Some referred to that behavior as they described in aggregate level terms why population would grow exponentially, often without making clear the relationship between individual and group-level behavior, and sometimes implicitly acknowledging the inconsistency between individual- and group-level behavior:

> **[Int 2]** ***even though it stays at .01*** **it seems it's becoming exponential.**
> **[Int 6] I don't know, but it is increasing exponentially, so uh, each person you said has a 1% chance of (mhm) reproducing, so like,** ***i dun, I don't know how to explain it.***
> **[Int 7] Um because at first they're starting out with 50 but eventually the,** ***even though there's a 1% chance those that do have children keep increasing the population*****, so it's only like natural that there would more towards the end, cause, because they have more opportunities to, to like, add more people to population.**

However, when explicitly prompted to explain whether a single individual in the population was more likely to reproduce as the rate of change of the population increased, most students came to the conclusion that individual behavior remained consistent throughout the run of the model, and that it was the fact that more people were added to the model as it ran that could enact that same reproductive behavior that increased the rate of change. In two cases, however, students had a good deal of difficulty in reconciling individual level and group level behavior. In both cases, the participants relied on their knowledge of how the exponential function worked in attempting to describe individual level behavior. In the first, Gary applies the exponential function to an individual's chance at reproducing. Furthermore, he applies it incorrectly; such that he comes to the conclusion that this chance decreases over time. When prompted by the interviewer to make sense of that in terms of "a person in the world" (the interviewer intended to refer to the modeled world; but is it likely she primed him to talk about real-world factors), he justifies the change in individual level behavior in terms of aging:

[Int 4]
M: So when you say they have more chance of reproducing if we're talking about that blue guy right there does he have more chance of reproducing?
S: He starts at 1%, right? (mhm) and it's 1% percent every tick isn't, then isn't it there's 1% every tick then for every tick that goes his chance like increases? or does it, like, I think, yeah.
M: Can you talk more about that?
S: If it's, can I write on?
M: Yeah, oh yeah, that's why there's paper here.
S: So there's, hold on, 1% chance for every tick (mhm) So for, if it's 1%, wait, let me think, hold on one sec.
M: Yeah that's fine, if you can say what you're thinking, too, you know (laughs)
S: So I'm trying to remember how I do this, if it's 1% probability per tick, over the span of five ticks, I think the probability increases, you multiply this, oh wait no, it decreases, I think. Cause it's .01 to the fifth power cause it's for every tick you multiply again by .01.
M: I see, so you're saying for like the blue guy, since, since each tick it's a .01 chance that for five ticks altogether it's---
S: .01 times .01 five times (Okay) Which is actually smaller then, yeah I think it's smaller.
M: Does that make sense?
S: Yeah
M: Okay, why is that? Like if you just think about a person in the world, you know?
S: Because as they get older, their uh reproduction system it like, it's not as healthy because it peaks at a certain point and then like, as you age, it becomes harder to produce (mmm) like you know, like reproduce.

Later, however, when asked to explain why the rate of change of the population is increasing, Gary begins to consider that individuals may not be changing their behavior, concluding “you also have to take into account that there's multiple people that have that .1 .01 percent chance”. As in Section 4.1.2, it seems that thinking about what generates rate of change prompted participants to consider behavior in a way that they were less likely to do when thinking only of the total population over time.

In the second case, Maria also adopts her understanding of they way the exponential function works, but to explain the behavior of the model as a whole. In other words, although she was told that the model simulated agent-level behavior and that each “person” had a 1% chance of reproducing (and asked again to confirm this during the beginning of the interview), she took this to mean that the value of the population was increased by 1% during each iteration. Because of this, she does not consider any individual-level behavior until she is confronted with the noise produced by the graph of people born per year:

[Int 9]
S: When is it changing the fastest? (yeah) Um, probably by the next time you go, it's gonna be the fastest, because there's um a bigger, a big population right now, so by next, 1%, let's say how many people there are right now, 1, 287, (mhm) is gonna be 2.87 people (okay) next time, so.
…
M: Mmkay. Does it make sense that it's as jaggedy as it is?
S: Yeh-uh...yeah, I think so, cause you can't have half a person or .8 person.
M: Mmkay. (so) So sometimes it's, it gets kind of lower and then it goes back up, does that make sense?
S: Well uh, I guess so, uh.
M: It's okay to say no. (laughs)
S: Um, cause, I don't know if this simulation is like, perfect like, cause people may not choose to have kids. (mhm) so. (okay) yeah. so some years there might be less and some years there might be more. (okay) but it evens out since it's an average (okay) 1.01%, er 1%.

In both of these cases, the participants heavily referenced their knowledge of the exponential function, and not behavior or graphs, to make sense of what was going on in the model. Later, when they began to think of complementary explanations and ways of representing change in the system, they were able to reconcile multiple levels in productive ways.

4.2.2 Understanding Birth as a Mechanism for Change

One of the most difficult aspects of the interview for students was making sense of the rate of change of a population as a measure of the number of born (in this specific model), or even as a count of change in the population more generally. Only one student explicitly used the plot of people born to help find and reason about the rate of change of the population, and many participants struggled with describing the relationship between the graphs of people born and total population. Many believed the plots to be different versions of the same quantity; with the smoother population graph representing an “edited” or “averaged” version of the plot of people born:

[Int 6] Um, so the population graph is more long term and the people born is more short term, like you can actually see how, um, like, like in depth the amount of people that are born or whatever each year but the population graph um it--you're just seeing like an increase, you don't see like the decreases as much.
[Int 10] I mean you could simply like we already related how this graph's irregular and so is this (mhm) but um, simply putting um, this isn't going to model dips in population nearly as well as this, (okay) so you could simply I guess say that this graph is simply almost like a best fit line of this graph so it takes like the top points are the most important pertinent points of the bottom graph (mhm) and it simply shows up on the top.
[Int 2] S: (pause) Well as there, as the population increases, so those does the number of people born.
M: Okay. Is this what you would expect to see for the number of people born?
S: Well overall
M: What do you mean by overall?
S: Like, looking at the best fit line. (Mhm) It's increasing (Mhm.) But, the um, they keep, like going from 0 to 12.1
[Int 3] I'm looking at (points to lower graph) this graph here (mhm) and it's just kind of edited version of that I guess (points to upper graph), when you're taking all of the like common, or I don't know, compounding all of them to make that same line.
M: Okay, can you talk about when you said edited, what that means.
S: In terms of edited with this? (Yeah.) Just like how like they had a point zero one chance of reproducing right? (mhm) So some aren't and some are (okay). Some are going to more than others. So when you compound it, it's going to come up with a line like that (points to upper graph), but the actual data might be (waves finger up and down) a (okay) lot more messy.

Irene did not think that there was a definite relationship between the two graphs, and viewed the number of individuals born as independent from the population. This is particularly interesting because it is true that in an agent-based model that includes elements of randomness, the exact number of individuals to be born during a given time unit cannot be determined before the time unit is executed. However, she was unable to be more specific about the relationship between the two graphs:

[Int 12]
S: Well, yes, I, there is a relation because the population depends on the number of people being born (mhm), but um, the people being, the, it's not vice versa, so, like the people being born is almost independent and the population depends on that, so.
M: Okay, almost independent how, can you talk a little more about that?

S: Um, because, you're um, people being born, isn't, is going to affect the population but, the number of people there isn't going to really affect the number of people being born. Yeah, I don't know if that makes sense.

Out of 11 interviews, 4 students explicitly referred to notions of the derivative and the reversibility of the population plot and people born plot and 4 noticed qualitatively the relationship between the plots. Often, this occurred after students noted their surprise that the plot of people born went both up and down; or after they were asked to explain why the plot was so noisy. We believe that this drew their attention to the perceptually related features of the graphs: namely, that high peaks on the graph of people born corresponded to “jumps” or “slants” in the graph of population, and that low valleys corresponded to a flattening of the graph.

One especially interesting episode occurred with Sarah, a student who was adept at explaining both the behavioral and mathematical reasoning behind the model, but had a good deal of trouble connecting those behavioral and mathematical explanations. When asked to interpret the rate of change of the population, she resorted to her understanding of the derivative as a measure of rate of change. Even when prompted to consider other ways to think about change “with all the information you’re given here”, it was not until she was confronted with the noisiness of the plot of people born, however, that she relates this notion of derivative to represent change in population as number of people born:

[Int 3]
M: Okay, and then one tick, to find the rate of change, what would you measure?
S: You could use derivatives. (laughs)
M: Mhm, is there any other way you could do it?
S: Uh, I dunno cause the problem with exponential functions using uh, solving for the slope in general, is that you come out with a straight line if you were to use it like you would solve for linear? (okay) Which isn't realistic for a population. I mean I guess it would be if you were only using one tick.
M: So, okay, can you think of any other way, with all the information you're given here, that you could do it?
S: To do what? To just...

M: To like, for a given tick, to say what the rate of change is.
S: Um, I dunno, I'd have to think about that. Kind of like derivatives all stuck in my mind. (laughs) (okay)
S: For a certain tick, um, well when I was using natural log functions we could always solve for, you know, the variable, we could solve for ticks or we could solve for the constant. Or the Y value I guess so you could make it into an equation and plug in exactly one tick and you're gonna get an exact number.

After noticing the noisiness of the graph of people born, Sarah considered that only a few individuals are reproducing each year. After that realization, we asked her to again explain the relationship between the two graphs:

S: Um (points at people born graph, then population graph) okay our original population is taking this (points to people born graph) added to uh people that there were, that there were beforehand, (mhm) before they were (mhm) the people were born. Um so it's taking in account to adding to the uh population beforehand (mhm), which is kind of the deal of exponents which is multiplying and multiplying and multiplyling (mhm) from the original
M: And so does that help you talk about rate of change at all?
S: In terms of population or in terms of...?
M: In terms of population.
S: People weren't, um, then I guess, oh, I guess this could be the rate of change.
M: And why's that?
S: Uh, because, well this, is it, yeah, because their rate of change is saying like oh, well this is how many people were added to the original population over a period of time
M: Mhm and how does that relate to kind of like the ideas you learned in class?
S: Um, about derivatives and stuff?
M: Yeah.
S: Um, that derivatives is basically taking like an exact (mhm) point divided by another exact point finding the exact um, like change, but this gives us the exact change over the exact time. It gives us the exact number of people born at a certain time which is what derivatives is, is solving for.

It is important to note that this is a high-performing classroom, and most students interviewed were very comfortable with the idea of the derivative and the relationship between the graph of a function and its derivative. Despite this, it seemed that students were not immediately able to recognize the plot of people born in the model as a representation of rate of change, or as related in a specific way to the population as a whole. Often, students began to explore this relationship more closely when they noticed or were confronted with the perceptual noisiness of the graph of people born: something that many of them found unexpected, and difficult to explain. Once these students noticed that the graph's noisiness was a result of randomness and heterogeneity, they quickly linked it to the population's reproduction behavior – which in turn, along with the now-meaningful graphical "noise" emphasizing related features of the two graphs, enabled them to identify people born as a measure of the rate of change of the population as a whole.

5. Discussion and Future Work

In this paper, and in our broader course of study, we are interested in mapping out the landscape of understanding of quantitative change in complex systems, how it relates to understanding quantitative change in motion-based and other systems, and how to foster such an understanding. We have introduced a preliminary framework that we hope will help us gain traction in identifying and studying the issues unique to change in complex systems – one that includes notions of randomness and heterogeneity, multiple behaviors, and multiple entities all as components and characteristics of quantitative change in systems. We have also studied how these notions come to bear on students' reasoning about a simple complex system: population growth modeled as an agent-based system, rather than with an exponential function. We found that while students did not have much difficulty explaining and reasoning about notions of rate of change and accumulation in this context, they employed a number of different (sometimes contradictory) resources in order to do so. They did, however, have trouble relating individual-level behaviors to the overall trends produced, as well as with identifying what *rate of change* represented specifically within the system.

We then presented a few specific cases of how these links were made. We argue that just as different representations emphasize different features of a situation, they also emphasize different *levels* of quantitative change: the behavior of individuals in the system, how those individuals can all simultaneously contribute to a change in that system, and how the system reflects those contributions over time. Our findings are consistent with other studies that note the relationship between representational choices and ways of reasoning; and the role of computation in those choices and reasoning (i.e. Parnafes, 2004; 2007). Below, we include a more general review of different concepts and information sources that are related to these different levels of quantitative change in systems.

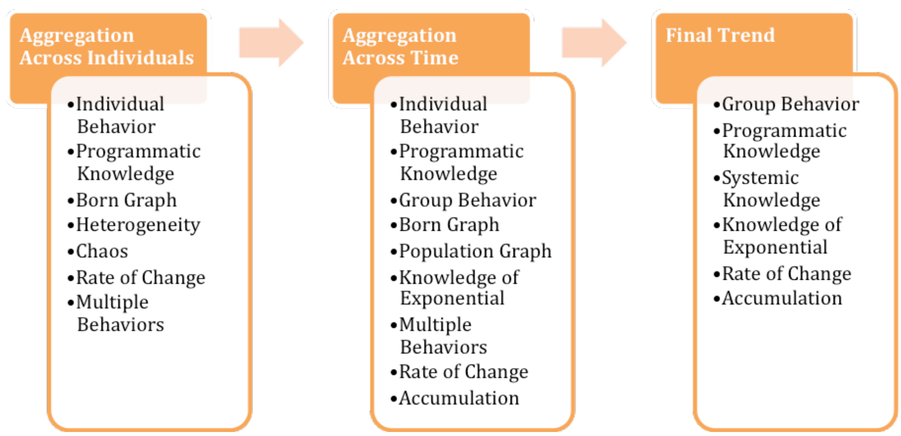


Figure 6. Different representations emphasize and enable students to relate different levels of change in complex systems

We argue that exposure to an agent-based treatment of complex systems; with the graphical noise, individual-level focus, and contextual grounding that it brings, can provide students with additional tools and information that can help them link individual behavior in a system to group level quantitative change, and group level quantitative change to overall change over time. We are currently developing a computational tool within the NetLogo (1999) agent-based modeling environment that allows students to quickly and easily assign domain-specific behaviors to agents within a simulated world, and compare and analyze in depth the quantitative patterns that result. We believe that by granting students the ability to program the simulations themselves; they were gain even more access to and grant even more attention to the individual-level behaviors and step-by-step temporal aspects (i.e., Sherin 2001) that we argue are not well represented by typical representations of quantitative change.

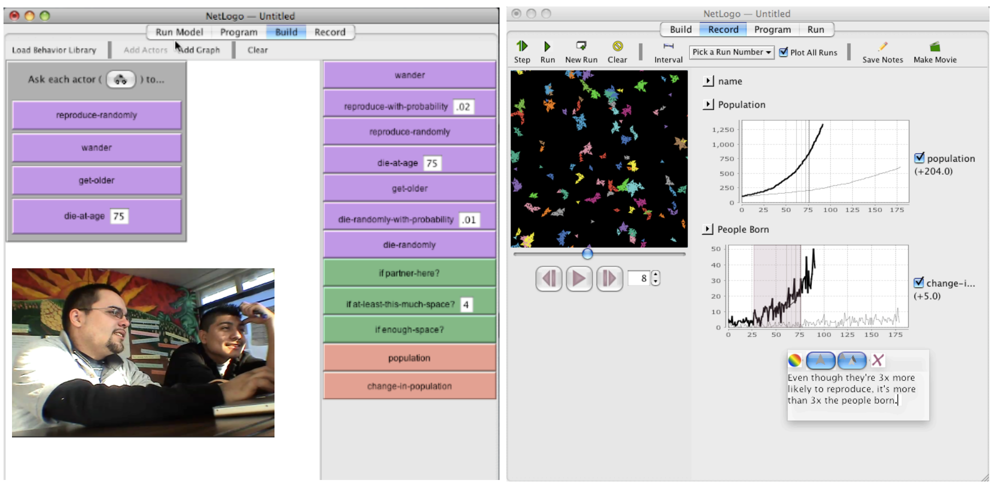


Figure 7. We are developing a simple programming language and analysis tools for students to model, explore, and interpret mathematical change in complex systems.

We have a number of new questions and investigations that will move this work forward. One question that emerged during our interactions with participants in this study was: what do students perceive as the "driving force" behind quantitative change – in mathematical models, computational models, and in the world? In the present study, some students identified time itself, or "ticks" of the model, as an independent variable as documented by Keene (2007) in her study of dynamical reasoning. Others suggested that it is the behavior of the entities that comprise a system, or rate of change itself, that creates quantitative change. It would be interesting to explore whether and how these perceptions affect one's reasoning about a phenomenon, and interpretation of representations of that change.

We are also interested in taking our new analytic framework and using it to begin to explore reasoning across different contexts, student populations, and symbolic encodings. We are currently interviewing students who have had much less opportunity to interact with these mathematical ideas at school than students from the present study, with a variety of representations of quantitative change (such as mathematical functions and system dynamics models) in addition to agent-based models. This work is feeding into the continued development of an educational environment that focuses on student development and analysis of agent-based models of systemic quantitative change.

6. Conclusion

Change is everywhere. It characterizes our world, and helps us to understand, interpret, and predict complex events and how they can influence our lives. Despite this, "conventional curricula neglect, delay, or deny students' access to [the mathematics of change and variation]" (Roschelle, et al., 2000) – and when students do learn about change, it is in the context of systems that are very unlike those that are most ubiquitous and meaningful in the real world. We hope that the development of these analytic tools and frameworks for better understanding *how* and *in what contexts* students consider mathematical change as it relates to large-scale systems, we can learn to provide all students with the tools to not only think, but also to act critically within their world.

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